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SENIOR MATHEMATICS COMPETITION 2013

Preliminary round Thursday
16th May 2013 Time
allowed 1 ½ hours

Instructions Attempt all questions. It is not expected that you will finish them all. Full working should accompany all solutions. Calculators may be used, but no other reference material is permitted. Total: 52 marks

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SENIOR MATHEMATICS COMPETITION 2013

1. A function f(x) obeys the rule $f(x+1) = f(x) - f(x-1) + 1$. Given that $f(2) = 8$ and $f(1) = 3$, find f(2013). [5]

2. Prove that $\sqrt{2}$ is irrational. [3]

3. For a Sierpinski Triangle (fractal : see diagram below) formed by successively bisecting sides of the shaded triangles, joining the bisection points and unshading the interior triangle, with S_0 having no unshaded triangles S_1 having one unshaded triangle (and, so, four non-overlapping interior triangles), show that the number of non-overlapping interior triangles for S_n is given by $(3^{n+1} - 1)/2$ [4]

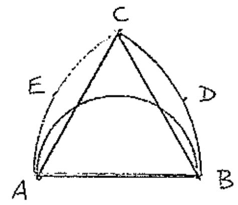
4. A disease test has a sensitivity of 0.95 (i.e. of those who test positive, 95% have the disease) and a specificity of 0.95 (of those who test negative, 95% do not have the disease).
(a) Show that if the true prevalence of the disease is 40% that over 90% of test results will be correct, and give the percentages of false positives and false negatives correct to 1d.p. [3]
(b) Show that if the true prevalence is 2%, that it is less likely that a laboratory positive result is correct, and give the percentages of false positives and negatives correct to 1d.p. [3]
(c) Make an appropriate generalisation about disease testing. [1]

5. Let $A = \sin(\pi - t) - \cos(\pi - t)$ and $B = \sin(\pi + t) - \cos(\pi + t)$. Find an expression for the difference $D = A - B$ in terms of sin(t) and give the maximum and minimum values this fluctuates between. [4]

6. For two numbers x and y, the arithmetic mean is $A=(x+y)/2$, the geometric mean is $G=\sqrt{(xy)}$ and the harmonic mean is $H=2/(1/x + 1/y)$. Prove that $\sqrt{(AH/2)} - \sqrt{2}\,G + G/\sqrt{2} = 0$ for all values of x and y. [3]

7. Given the function $f(n) = 2^n + (n-1)(n-2)(n-3)(n-4)x$, where $n \in N$ and $x \in R$, explain why the fifth number in the sequence 2,4,8,16 ... could be any real number, and then determine the value of f(5) if $x = -1/24$ [2]

8. An equilateral triangle ACB is drawn, with arcs AEC and BDC each of radius AB. (See diagram on next page)
Find the ratio Area of semicircle on AB: Area AECDB. [3]



9. Rotten teeth: A person (with 32 teeth) is offered the deal of $1million to remove the whole lot, or 1 cent for the first, 2 cents for the second, 4 cents for the third, 8 cents for the fourth and so on.
(a) Calculate the exact cost for the second option.
(b) If several dentists each extracted fewer than 32 teeth, all using the second cost option, the cost could be as little as $0.32; but the bus fare from one dentist to the next is a flat rate of $2, giving a maximum cost of $64.32. Determine the number of dentists and bus rides to give the least cost, and the amount of that cost. [4]

10. Given that the approximate radius of the earth is 6378km, and assuming (incorrectly) that the earth is approximately spherical, show that the distance to the horizon from a point at elevation H is given by D(miles) = $1.2\sqrt{H}$ (feet). Note: 1 mile = 1.62km, 1mile = 5280 feet. [4]

11. If $a*b = a/b + b/a$, find an expression for $(a-b) * (a+b)$ [2]

12. Four aces are face down side by side. Two are chosen at random and turned over. What is the probability they are different colours, and why? [2]

13. If $f(2x^2 - 1) = 2xf(x)$ for all values of x, and $g(t) = [f(\cos(t))]/\sin(t)$, show that
(a) $g(t+\pi) = g(t)$ and [2]
(b) $g(2t) = g(t)$ [2]

14. For the equation $x^2 - 12x + r = 0$, if there are two rational roots how many possible values of r are there? Find three of those values and then characterize all the possible values (ie. What features of numbers do you notice?) [5]

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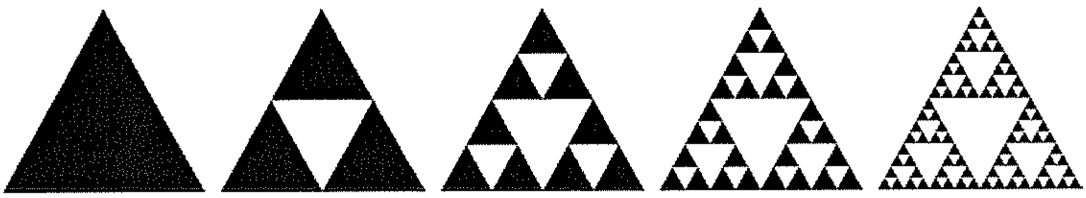
1. A function f(x) obeys the rule f(x+1) = f(x) – f(x-1) + 1. Given that f(2) = 8 and f(1) = 3, find f(2013). [5]

Pattern: 3,8,6,-1,-6,-4,3 repeats √√√ then 2013/6 has rem 3 √ so f(2013) = 6 √

2. Prove that √2 is irrational. [3]

Let sqrt 2 = a/b, where a and b are mutually prime, √ square √ and get contradiction. √

3. For a Sierpinski Triangle (fractal : see diagram below) formed by successively bisecting sides of the shaded triangles, joining the bisection points and unshading the interior triangle, with S_0 having no unshaded triangles, S_1 having one unshaded triangle (and, so, four non-overlapping interior triangles), show that the number of non-overlapping interior triangles for S_n is given by $(3^{n+1} – 1)/2$. S_1 has 4 = 3 + 1, S_2 has 13 =3^2 + 3 + 1, S_3 has 40 √ sequence of powers of 3, plus 1. √ Use Sum of geometric sequence, √ simplify to get result. √ [4]



4. A disease test has a sensitivity of 0.95 (i.e. of those who test positive, 95% have the disease) and a specificity of 0.95 (of those who test negative, 95% do not have the disease).

(a) Show that if the true prevalence of the disease is 40% that over 90% of test results will be correct, and give the percentages of false positives and false negatives correct to 1d.p. Table, √ 7.3%, √ 3.4% √ [3]

(b) Show that if the true prevalence is 2%, that it is less likely that a laboratory positive result is correct, and give the percentages of false positives and negatives correct to 1d.p. Table, √ 72.1%,√ 0.1% √ [3]

(c) Make a generalisation about disease testing. Low prevalence implies less likely lab positive correct. √ [1]

5. Let A = sin(π – t) – cos(π – t) and B = sin(π + t) – cos(π + t) . Find an expression for the difference D = A - B in terms of sin(t) and give the maximum and minimum values this fluctuates between. [4]

A=sint+cost. √ B=cost-sint. √ D=2sint: √ values are -2 to 2. √

6. For two numbers x and y, the arithmetic mean is A=(x+y)/2, the geometric mean is G=√(xy) and the harmonic mean is H=2/(1/x + 1/y). Prove that √(AH/2) - √2 G + G/√2 = 0 for all values of x and y. [3]

LHS=√(xy/2)- √2√(xy)+ √(xy)/ √2 √= simplifies √ = RHS √

7. Given the function f(n) = 2^n + (n-1)(n-2)(n-3)(n-4)x, where n ε N and x ε R, explain why the fifth number in the sequence 2,4,8,16 ... could be any real number, and then determine the value of f(5) if x = -1/24 [2]

5^{th} number is 2^5+24x, choose x to get any number; √ f(5) = 31 √

8. An equilateral triangle ACB is drawn, with arcs AEC and BDC each of radius AB. Find the ratio Area of semicircle on AB: Area AECDB. [3]

Areas of sector, triangle, segment, semicircle; numerical or algebraic √√ give 3 : 8 - 6√3/π. √ Accept 3 π: (8 π - 6√3)

9. Rotten teeth: A person (with 32 teeth) is offered the deal of \$1million to remove the whole lot, or 1 cent for the first, 2 cents for the second, 4 cents for the third, 8 cents for the fourth and so on.

(a) Calculate the exact cost for the second option. \$42,949,672.95 √

(b) If several dentists each extracted fewer than 32 teeth, all using the second cost option, the cost could be as little as \$0.32; but the bus fare from one dentist to the next is a flat rate of \$2, giving a maximum cost of \$64.32. Determine the number of dentists and bus rides to give the least cost, and the amount of that cost. [4]

8 dentists and7 rides; √√ \$15.20 √

10. Given that the approximate radius of the earth is 6378km, and assuming (incorrectly) that the earth is approximately spherical, show that the distance to the horizon from a point at elevation H is given by D(miles) = 1.2√H (feet). Note: 1 mile = 1.62km, 1mile = 5280 feet. [4]

$D^2=(r+H)^2 - r^2$ √ ~Hx2r √ since r>>H; so D(miles) = √(2r) √(H/5280) √ ~1.23√H as required. √

11. If a*b = a/b + b/a, find an expression for (a-b) * (a+b) [2]

$2(a^2+b^2)/(a^2 - b^2)$ √√

12. Four aces are face down side by side. Two are chosen at random and turned over. What is the probability they are different colours, and why? [2]

2/3; √ because – show table. √

13. If $f(2x^2 - 1) = 2xf(x)$ for all values of x, and g(t) =[f(cos(t))]/sin(t), show that

(a) g(t+ π) = g(t) and f(-cost)/-(sint) , f odd function; √√ [2]

(b) g(2t) = g(t) $f(2\cos^2 t - 1)/\sin(2t)=2\cos t f(\cos t)/2\sin t\cos t$ √√ [2]

14. For the equation $x^2 - 12x + k = 0$, if there are two rational roots how many possible values of k are there? Find three of those values and then characterize all the possible values (ie. What features of numbers do you notice?) [5]

11 values, √ 35.75, 35, 33.75, 29.75, 27, 23.75, 10, 15.75, 11, 5.75, 0. √√√ All whole or whole + 0.75 √