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SENIOR MATHEMATICS COMPETITION 2014

Preliminary round Thursday 22nd May 2014 Time allowed 1 ½ hours

Instructions Attempt all questions. It is not expected that you will finish them all. Full working should accompany all solutions. Calculators may be used, but no other reference material is permitted. Total: 50 marks

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1. (a) Find values for A and B so that for all appropriate values of y

$\frac{A}{y} + \frac{B}{b-y} = \frac{b}{y(b-y)}$, $b \neq 0$ A = 1 √ B = 1 √

(b) What is the domain of $f(y) = \frac{b}{y(b-y)}$? All real numbers except 0 and b. √ [3]

2. Find $\lim_{x \to \infty} \frac{x^2}{e^x}$ e^x increases faster than x^2 √ so limit = 0 √ [2]

3. If television channels broadcast 15 minutes of commercials every hour, and for any given channel the timing of the start of a commercial break can be considered random, then how many channels need to be broadcasting at once for a viewer to be reasonably sure (more than 99% sure) of being able to avoid commercials by switching channels? Let p = 0.75, q = 0.25, one channel P(success) = 0.75, two: P = 0.938, three: P = 0.984, √ four: P = 0.996; so four channels. √ √ [3]

4. Sacks of rice are weighed together. Sacks 1 and 2 together weigh 12kg, sacks 2 and 3 together weigh 13.5kg, sacks 3 and 4 together weigh 11.5kg, sacks 4 and 5 together weigh 8kg, and sacks 1, 3 and 5 together weigh 16kg. What does each sack weigh? Sum all, subtract twice the sum of sacks 1,2,4 and 5. √√ S1 = 5.5, S2 = 6.5, S3 = 7, S4 = 4.5, S5 = 3.5 all kg √ [3]

5. A rectangular garden, half a metre longer than it is wide, consists entirely of a gravel walk, spirally arranged, one metre wide and 3630 metres long. Find the dimensions of the garden. x(x+0.5) = 3630 solve since number of m in walking straight = area in m^2 and the distance in one m^2 at a corner is 1m. √ 60 by 60.5m √ [2]

6. When a whole number x is divided by three the remainder is two, when x is divided by five, the remainder is three, and when x is divided by seven, the remainder is two. Find:

(a) the smallest value for x and 23 √

(b) the general solution for x. 3 x 5 x 7 = 105, so any number 23 + 105n for integral n. √√ [3]

7. My graphics calculator gives the value of the square root of two as 1.414213562373 to 12 decimal places. What are the next two digits and why? if x = value, $(x + a)^2 = 2$, a = 9.50488×10^{-14} √ so 0 then 9 √ [2]

8. A goat is tethered to the corner of a shed which is 2m long and 2m wide with a rope that is 4.5m long. What is the maximum area the goat can graze? ¾ circle r = 4.5√ 2x ¼ circle r = 2.5 √ subtract overlap which is 2x shape with two straight sides and one curved, found by sector side 2.5, angle 0.184 radians√ minus triangle √ 57.3sq m √ [5]

9. The Ackermann – Peter function has the properties

$$A(m, n) = \begin{cases} n + 1 & \text{if } m = 0 \\ A(m-1, 1) & \text{if } m > 0 \text{ and } n = 0 \\ A(m-1, A(m, n-1)) & \text{if } m > 0 \text{ and } n > 0 \end{cases}$$

so that A(0, 0) = 1; A(1, 1) = 3; A(2, 2) = 7

(a) find the value of A(3, 3) 61 √ (b) find an expression involving n for A(3, n) $2^{n+3} - 3$ √

(c) find the value of A(4, 1) 65533 √ (d) find an expression involving whole numbers for A(4, 4)

2^2^2^2^2^2^2 – 3 √√ [5]

10. (a) If sin (ϴ) tan (ϴ) + cos (ϴ) = $\frac{1}{\tan(\theta)}$ show that

Sin ^{2}(ϴ) + sin (ϴ) – 1 = 0 LHS = $\frac{1}{\cos(\theta)}$ by Pythagorean identity, √√ so sin (ϴ) = cos ^{2}(ϴ), and use identity again to get result. √ [3]

(b) Hence solve sin (ϴ) tan (ϴ) + cos (ϴ) = $\frac{1}{\tan(\theta)}$ for $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

and show that ϴ is between $\frac{\pi}{6}$ and $\frac{\pi}{4}$. sin (ϴ) = 0.618... or -1.618...√ (negative not possible); √ and since sin$\frac{\pi}{6}$ = 0.5 and sin$\frac{\pi}{4}$ = 0.7071 then ϴ is between $\frac{\pi}{6}$ and $\frac{\pi}{4}$. √ [3]

11. P is any point inside rectangle ABCD. Prove that $PA^2 + PC^2 = PB^2 + PD^2$. Draw vertical XPZ and horizontal YP through P, and use Pythagoras: $AP^2 = ZP^2 + AZ^2$; √ $CP^2 = PX^2 + XC^2$; so $AP^2 + CP^2 = ZP^2 + AZ^2 + PX^2 + XC^2$; √ similarly $PB^2 + PD^2 = PX^2 + BX^2 + ZP^2 + ZD^2 = PX^2 + AZ^2 + ZP^2 + AZ^2 = PA^2 + PC^2$ as required √ [3]

12. In 1987 in the USA the data on HIV and on testing gave the following results:
The probability that a patient with HIV tests positive for HIV is 0.98
The probability that a person without HIV tests negative is 0.93
The proportion of people in the population infected with HIV is 0.01

(a) Complete the following table using the data given, and
(b) Give the probability that a randomly chosen member of the population tests positive for HIV.
(c) What proportion of people testing positively actually have HIV?

Probability table	Positive test result	Negative test result	Total
Has HIV	0.0098	0.0002	0.01
Does not have HIV	0.0693	0.9207	0.99
Total	0.0791	0.9209	1

(a) Answers in blue √ The rest of the answers √ (b) P(tests positive) = 0.0791 √
(c) P(HIV\pos) = 0.0098/0.0791 = 0.124 i.e. 12.4% √ [4]

13. At the Board meeting of a business it was noted that the business had made a profit over every consecutive eight month period and a loss over every consecutive five month period since the previous meeting. What is the maximum possible number of months since the previous meeting? 5, -8, 5, 5, -8, 5, -8, 5, 5, -8, 5. √ cannot use 5 next, or -8 next √ so 11 months. √ [3]

14. For functions f: x ⟶ $\frac{2}{x+1}$ and g: x ⟶ $\frac{3}{x}$

(a) Find a formula for the composite function f o g (x) = f(g(x)) y = $\frac{2x}{3+x}$ √

(b) Sketch a labelled graph of the inverse function f o g $^{-1}$(x) working: 2y = x(3 + y) √ y = $\frac{3x}{2-x}$ √
asymptotes x=2, y=-3, intercept (0,0) √ [3]

15. A kangaroo makes 10 consecutive jumps, each one metre, each in the direction N, S, E or W. How many different positions? Diagram and/or working √√ n=10, $(n+1)^2 = 121$ √ [3]